

PARISIAN RUIN OF THE BROWNIAN MOTION RISK MODEL WITH CONSTANT FORCE OF INTEREST

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Abstract: Let $B(t), t \in \mathbb{R}$ be a standard Brownian motion. Define a risk process

$$(0.1) \quad R_u^\delta(t) = e^{\delta t} \left(u + c \int_0^t e^{-\delta s} ds - \sigma \int_0^t e^{-\delta s} dB(s) \right), t \geq 0,$$

where $u \geq 0$ is the initial reserve, $\delta \geq 0$ is the force of interest, $c > 0$ is the rate of premium and $\sigma > 0$ is a volatility factor. In this contribution we obtain an approximation of the Parisian ruin probability

$$\mathcal{K}_S^\delta(u, T_u) := \mathbb{P} \left\{ \inf_{t \in [0, S]} \sup_{s \in [t, t+T_u]} R_u^\delta(s) < 0 \right\}, S \geq 0,$$

as $u \rightarrow \infty$ where T_u is a bounded function. Further, we show that the Parisian ruin time of this risk process can be approximated by an exponential random variable. Our results are new even for the classical ruin probability and ruin time which correspond to $T_u \equiv 0$ in the Parisian setting.

Key Words: Parisian ruin; ruin probability; ruin time; Brownian motion

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1. INTRODUCTION

In a theoretical insurance model the surplus process $R_u(t)$ can be defined by

$$R_u(t) = u + ct - X(t), \quad t \geq 0,$$

see [10], where $u \geq 0$ is the initial reserve, $c > 0$ is the rate of premium and $X(t), t \geq 0$ denotes the aggregate claims process. More specifically, we assume that the aggregate claims process is a Brownian motion, i.e., $X(t) = \sigma B(t)$, $\sigma > 0$. Due to the nature of the financial market, we shall consider a more general surplus process including interest rate, see [18], called a risk reserve process with constant force of interest, i.e., $R_u^\delta(t), t \geq 0$, in (0.1). See [18, 3, 14] for more studies on risk models with force of interest.

During the time horizon $[0, S], S \in (0, \infty]$, the classical ruin probability is defined as below

$$(1.1) \quad \psi_S^\delta(u) := \mathbb{P} \left\{ \inf_{t \in [0, S]} R_u^\delta(t) < 0 \right\},$$

see [10, 15, 16, 8]. In [9, 11] the exact formula of $\psi_\infty^\delta(u)$ for $\delta > 0$ is shown to be

$$\psi_\infty^\delta(u) = \frac{\Psi \left(\sqrt{\frac{2\delta}{\sigma^2}} u + \sqrt{\frac{2c^2}{\sigma^2\delta}} \right)}{\Psi \left(\sqrt{\frac{2c^2}{\sigma^2\delta}} \right)}, \quad u > 0,$$

where $\Psi(x) = 1 - \Phi(x)$ with $\Phi(\cdot)$ the distribution function of an $\mathcal{N}(0, 1)$ random variable.

For $\delta = 0$, the exact value of $\psi_\infty^0(u)$ is well-known (cf. [7]) with

$$\psi_\infty^0(u) = e^{-\frac{2cu}{\sigma^2}}, \quad u > 0.$$

In the literature, there are no results for the classical ruin probability in the case of finite time horizon, i.e., $S \in (0, \infty)$. For $S \in (0, \infty)$, with motivation from the recent contributions [4, 5] we shall investigate in this paper the Parisian ruin probability over the time period $[0, S]$ defined as

$$(1.2) \quad \mathcal{K}_S^\delta(u, T_u) := \mathbb{P} \left\{ \inf_{t \in [0, S]} \sup_{s \in [t, t+T_u]} R_u^\delta(s) < 0 \right\},$$

where $T_u \geq 0$ models the pre-specified time. Our assumption on T_u is that

$$\lim_{u \rightarrow \infty} T_u u^2 = T \in [0, \infty)$$

and thus $\psi_S^\delta(u)$ is a special case of $\mathcal{K}_S^\delta(u, T_u)$ with $T_u \equiv 0$.

Another quantity of interest is the conditional distribution of the ruin time for the surplus process $R_u^\delta(t)$. The classical ruin time, e.g., [3, 12, 16], is defined as

$$(1.3) \quad \tau(u) = \inf\{t > 0 : R_u^\delta(t) < 0\}.$$

Here as in [4] we define the Parisian ruin time of the risk process $R_u^\delta(t)$ by

$$(1.4) \quad \eta(u) = \inf\{t \geq T_u : t - \kappa_{t,u} \geq T_u, R_u^\delta(t) < 0\}, \quad \text{with } \kappa_{t,u} = \sup\{s \in [0, t] : R_u^\delta(s) \geq 0\},$$

and $\tau(u)$ is a special case of $\eta(u)$ with $T_u \equiv 0$.

Brief organization of the rest of the paper: In Section 2 we first present our main results on the asymptotics of $\mathcal{K}_S^\delta(u, T_u)$ as $u \rightarrow \infty$ and then we display the approximation of the Parisian ruin time. All the proofs are relegated to Section 3.

2. MAIN RESULTS

Before giving the main results, we shall introduce a generalized Piterbarg constant as

$$(2.1) \quad \tilde{\mathcal{P}}(T) = \lim_{\lambda \rightarrow \infty} \tilde{\mathcal{P}}(\lambda, T), \quad T \geq 0,$$

where for $\lambda, T \geq 0$

$$\tilde{\mathcal{P}}(\lambda, T) = \mathbb{E} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, T]} e^{\sqrt{2}B(t-s) - |t-s| - (t-s)} \right\}.$$

Note further that the classical Piterbarg constant $\mathcal{P}_1^1[0, \infty)$ equals $\tilde{\mathcal{P}}(0)$ and $\mathcal{P}_1^1[0, \infty) = 2$, see [6, 1, 13].

Through this paper \sim means asymptotic equivalence when the argument tends to 0 or ∞ . Recall that $\Psi(\cdot)$ denotes the tail distribution function of an $\mathcal{N}(0, 1)$ random variable and $\Psi(u) \sim \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}, u \rightarrow \infty$.

Theorem 2.1. *For $\delta > 0, S > 0$ and $\lim_{u \rightarrow \infty} T_u u^2 = T \in [0, \infty)$, we have*

$$(2.2) \quad \mathcal{K}_S^\delta(u, T_u) \sim \tilde{\mathcal{P}}(aT) \Psi \left(\frac{\sqrt{2\delta}(u + \frac{c}{\delta}(1 - e^{-\delta S}))}{\sigma \sqrt{1 - e^{-2\delta S}}} \right), \quad u \rightarrow \infty,$$

where $a := \frac{2\delta^2 e^{-2\delta S}}{\sigma^2(1 - e^{-2\delta S})^2}$.

Remarks 2.2. *a) When $T_u \equiv 0$, $\mathcal{K}_S^\delta(u, T_u)$ reduces to the classical ruin probability $\psi_S^\delta(u)$, and by Theorem 2.1 with $T = 0$*

$$\mathcal{K}_S^\delta(u, 0) = \psi_S^\delta(u) \sim 2\Psi \left(\frac{\sqrt{2\delta}(u + \frac{c}{\delta}(1 - e^{-\delta S}))}{\sigma \sqrt{1 - e^{-2\delta S}}} \right), \quad u \rightarrow \infty.$$

b) If $\delta = 0$

$$(2.3) \quad \begin{aligned} \mathcal{K}_S^0(u, T_u) &= \mathbb{P} \left\{ \inf_{t \in [0, S]} \sup_{s \in [t, t+T_u]} (u + cs - \sigma B(s)) < 0 \right\} \\ &\sim \tilde{\mathcal{P}}(bT) \Psi \left(\frac{u + cS}{\sigma \sqrt{S}} \right), \quad u \rightarrow \infty, \end{aligned}$$

where $b := \frac{1}{2\sigma^2 S^2}$ and we used the result of Corollary 3.4 (ii) in [5].

Further, if $\delta = 0$ and $T_u \equiv 0$, by (2.3) with $T = 0$, we get the asymptotic result of the classical ruin probability

$$(2.4) \quad \psi_S^0(u) \sim 2\Psi \left(\frac{u + cS}{\sigma \sqrt{S}} \right), \quad u \rightarrow \infty.$$

In fact, [7] gave the exact result of $\psi_S^0(u)$, $u > 0$, i.e.,

$$\begin{aligned} \psi_S^0(u) &= \Psi \left(\frac{u + cS}{\sigma \sqrt{S}} \right) + e^{-\frac{2cu}{\sigma^2}} \Phi \left(\frac{cS - u}{\sigma \sqrt{S}} \right) \\ &\sim 2\Psi \left(\frac{u + cS}{\sigma \sqrt{S}} \right), \quad u \rightarrow \infty, \end{aligned}$$

which follows from

$$\lim_{u \rightarrow \infty} \frac{e^{-\frac{2cu}{\sigma^2}} \Phi \left(\frac{cS - u}{\sigma \sqrt{S}} \right)}{\Psi \left(\frac{u + cS}{\sigma \sqrt{S}} \right)} = \lim_{u \rightarrow \infty} \frac{-\frac{2c}{\sigma^2} e^{-\frac{2cu}{\sigma^2}} \Phi \left(\frac{cS - u}{\sigma \sqrt{S}} \right) - \frac{1}{\sigma \sqrt{2\pi S}} e^{-\left(\frac{u + cS}{\sigma \sqrt{S}}\right)^2/2}}{-\frac{1}{\sigma \sqrt{2\pi S}} e^{-\left(\frac{u + cS}{\sigma \sqrt{S}}\right)^2/2}} = 1.$$

Our next result discusses the approximation of the conditional ruin time.

Theorem 2.3. Let $\eta(u)$ satisfy (1.4), under the assumptions of Theorem 2.1, we have for any $x > 0$ and $\delta \geq 0$,

$$(2.5) \quad \mathbb{P} \{ u^2(S + T_u - \eta(u)) > x \mid \eta(u) \leq S + T_u \} \sim \begin{cases} \exp(-ax), & \text{if } \delta > 0, \\ \exp(-bx), & \text{if } \delta = 0, \end{cases} \quad u \rightarrow \infty,$$

where $a := \frac{2\delta^2 e^{-2\delta S}}{\sigma^2(1 - e^{-2\delta S})^2}$ and $b := \frac{1}{2\sigma^2 S^2}$.

Remark 2.4. If $T_u \equiv 0$, then $\eta(u) = \tau(u)$ and by Theorem 2.3, we obtain as $u \rightarrow \infty$

$$\mathbb{P} \{ u^2(S - \tau(u)) > x \mid \tau(u) \leq S \} \sim \begin{cases} \exp(-ax), & \text{if } \delta > 0, \\ \exp(-bx), & \text{if } \delta = 0. \end{cases}$$

3. PROOFS

Hereafter we assume that $\mathbb{C}_i, i \in \mathbb{N}$ are some positive constants.

PROOF OF THEOREM 2.1 For $S > 0$ and u large enough

$$\begin{aligned} \mathcal{K}_S^\delta(u, T_u) &= \mathbb{P} \left\{ \sup_{t \in [0, S]} \inf_{s \in [t, t+T_u]} \left(\sigma \int_0^s e^{-\delta z} dB(z) - c \int_0^s e^{-\delta z} dz \right) > u \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in [0, S]} \inf_{s \in [t, t+T_u]} \overline{X}(s) \frac{f_u(S)}{f_u(s)} > f_u(S) \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in [0, S]} \inf_{s \in [t, t+T_u]} X_u(s) > f_u(S) \right\}, \end{aligned}$$

with

$$X(s) = \sigma \int_0^s e^{-\delta z} dB(z), \quad \overline{X}(s) = \frac{X(s)}{\sigma_X(s)}, \quad f_u(s) = \frac{u + \frac{c}{\delta}(1 - e^{-\delta s})}{\sigma_X(s)} \quad \text{and} \quad X_u(s) = \overline{X}(s) \frac{f_u(S)}{f_u(s)},$$

where $\sigma_X^2(s)$ is the variance of $X(s)$ with $\sigma_X^2(s) = \frac{\sigma^2}{2\delta}(1 - e^{-2\delta s})$.

Set $\rho(u) = \left(\frac{\ln u}{u}\right)^2$ and for any $\lambda > 0$, Bonferroni inequality yields

$$(3.1) \quad \Pi_0(u) := \mathbb{P} \left\{ \sup_{t \in [S - \lambda u^{-2}, S]} \inf_{s \in [t, t + T_u]} X_u(s) > f_u(S) \right\} \leq \mathcal{K}_S^\delta(u, T_u) \leq \Pi_0(u) + \Pi_1(u) + \Pi_2(u),$$

where

$$\Pi_1(u) = \mathbb{P} \left\{ \sup_{t \in [0, S - \rho(u)]} \inf_{s \in [t, t + T_u]} X_u(s) > f_u(S) \right\}, \quad \Pi_2(u) = \mathbb{P} \left\{ \sup_{t \in [S - \rho(u), S - \lambda u^{-2}]} \inf_{s \in [t, t + T_u]} X_u(s) > f_u(S) \right\}.$$

First we give some upper bounds of $\Pi_1(u)$ and $\Pi_2(u)$ which finally show that

$$(3.2) \quad \Pi_1(u) + \Pi_2(u) = o(\Pi_0(u)), \quad u \rightarrow \infty.$$

For all u large

$$\begin{aligned} \mathbb{E} \{ (X_u(t_1) - X_u(t_2))^2 \} &= \mathbb{E} \left\{ \left(X(t_1) \frac{f_u(S)}{u + \frac{c}{\delta}(1 - e^{-\delta t_1})} - X(t_2) \frac{f_u(S)}{u + \frac{c}{\delta}(1 - e^{-\delta t_2})} \right)^2 \right\} \\ &\leq \mathbb{C}_1 \mathbb{E} \left\{ \left(\int_{t_1}^{t_2} e^{-\delta z} dB(z) \right)^2 \right\} + \mathbb{C}_2 \left(\frac{u + \frac{c}{\delta}(1 - e^{-\delta S})}{u + \frac{c}{\delta}(1 - e^{-\delta t_1})} - \frac{u + \frac{c}{\delta}(1 - e^{-\delta S})}{u + \frac{c}{\delta}(1 - e^{-\delta t_2})} \right)^2 \\ &\leq \mathbb{C}_3 |t_1 - t_2|, \quad t_1 < t_2, \quad t_1, t_2 \in (0, S]. \end{aligned}$$

Moreover,

$$\sup_{t \in [0, S - \rho(u)]} \text{Var}(X_u(t)) = \sup_{t \in [0, S - \rho(u)]} \left(\frac{f_u(S)}{f_u(t)} \right)^2 = \frac{f_u^2(S)}{f_u^2(S - \rho(u))},$$

where we use the fact that $f_u(t)$ is a decreasing function for $t \in [0, S]$ when u large enough. Therefore, by Theorem 8.1 in [17], we obtain

$$(3.3) \quad \Pi_1(u) \leq \mathbb{P} \left\{ \sup_{t \in [0, S - \rho(u)]} X_u(t) > f_u(S) \right\} \leq \mathbb{C}_4 u^2 \Psi(f_u(S - \rho(u))),$$

and direct calculation yields that

$$\begin{aligned} u^2 \Psi(f_u(S - \rho(u))) &\leq \frac{u^2}{\sqrt{2\pi} f_u(S)} e^{-\frac{f_u^2(S)}{2}} \left(\frac{f_u^2(S - \rho(u))}{f_u^2(S)} - 1 \right) e^{-\frac{f_u^2(S)}{2}} \\ &\sim u^2 e^{-a(\ln u)^2} \Psi(f_u(S)) = o(\Psi(f_u(S))), \quad u \rightarrow \infty, \end{aligned}$$

where $a = \frac{2\delta^2 e^{-2\delta S}}{\sigma^2(1 - e^{-2\delta S})^2}$ and we use the fact that

$$(3.4) \quad 1 - \frac{f_u(S)}{f_u(S - t)} \sim \frac{\delta e^{-2\delta S}}{1 - e^{-2\delta S}} t, \quad t \rightarrow 0.$$

Set

$$\Delta_k = [k\lambda u^{-2}, (k+1)\lambda u^{-2}], \quad k \in \mathbb{N}, \quad \text{and} \quad N(u) = \lceil \lambda^{-1} \rho(u) u^2 \rceil,$$

where $\lceil \cdot \rceil$ stands for the ceiling function, then

$$\begin{aligned} \Pi_2(u) &\leq \mathbb{P} \left\{ \sup_{t \in [S - \rho(u), S - \lambda u^{-2}]} X_u(t) > f_u(S) \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in [\lambda u^{-2}, \rho(u)]} X_u(S - t) > f_u(S) \right\} \\ &\leq \sum_{k=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in \Delta_k} X_u(S - t) > f_u(S) \right\} \end{aligned}$$

$$\begin{aligned}
(3.5) \quad & \leq \sum_{k=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in \Delta_0} \bar{X}(S-t) > f_u(S - k\lambda u^{-2}) \right\} \\
& \leq \sum_{k=1}^{N(u)} \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \bar{X}(S - u^{-2}t) > f_u(S - k\lambda u^{-2}) \right\}.
\end{aligned}$$

Clearly,

$$(3.6) \quad \inf_{1 \leq k \leq N(u)} f_u(S - k\lambda u^{-2}) \rightarrow \infty, u \rightarrow \infty,$$

and for $t_1 < t_2$, $t_1, t_2 \in [0, S]$,

$$r_X(t_1, t_2) := \mathbb{E} \{ \bar{X}(t_1) \bar{X}(t_2) \} = \sqrt{\frac{1 - e^{-2\delta t_1}}{1 - e^{-2\delta t_2}}}.$$

Further,

$$\begin{aligned}
(3.7) \quad & \lim_{u \rightarrow \infty} \sup_{1 \leq k \leq N(u)} \sup_{\substack{t_1 \neq t_2, \\ t_1, t_2 \in [0, \lambda]}} \left| f_u^2(S - k\lambda u^{-2}) \frac{\text{Var}(\bar{X}(S - u^{-2}t_1) - \bar{X}(S - u^{-2}t_2))}{2a|t_1 - t_2|} - 1 \right| \\
& = \lim_{u \rightarrow \infty} \sup_{1 \leq k \leq N(u)} \sup_{\substack{t_1 \neq t_2, \\ t_1, t_2 \in [0, \lambda]}} \left| f_u^2(S - k\lambda u^{-2}) \frac{2 - 2r_X(S - u^{-2}t_1, S - u^{-2}t_2)}{2a|t_1 - t_2|} - 1 \right| \\
& = 0,
\end{aligned}$$

and

$$\begin{aligned}
(3.8) \quad & \sup_{1 \leq k \leq N(u)} \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} f_u^2(S - k\lambda u^{-2}) \mathbb{E} \{ (\bar{X}(S - u^{-2}t_1) - \bar{X}(S - u^{-2}t_2)) \bar{X}(S) \} \\
& \leq \mathbb{C}_5 u^2 \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} |r_X(S - u^{-2}t_1, S) - r_X(S - u^{-2}t_2, S)| \\
& \leq \mathbb{C}_6 u^2 \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} \left| \sqrt{1 - e^{-2\delta(S - u^{-2}t_1)}} - \sqrt{1 - e^{-2\delta(S - u^{-2}t_2)}} \right| \\
& \leq \mathbb{C}_7 \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} |t_1 - t_2| \rightarrow 0, \quad u \rightarrow \infty, \varepsilon \rightarrow 0.
\end{aligned}$$

According to (3.6), (3.7), (3.8) and Lemma 5.3 of [2], (3.5) is followed by

$$(3.9) \quad \Pi_2(u) \leq \mathbb{C}_8 \lambda \sum_{k=1}^{N(u)} \Psi(f_u(S - k\lambda u^{-2})) \leq \mathbb{C}_9 \Psi(f_u(S)) \lambda \sum_{k=1}^{\infty} e^{-\mathbb{C}_{10} k \lambda} = o(\Psi(f_u(S))), \quad u \rightarrow \infty, \lambda \rightarrow \infty,$$

where the last inequality follows from (3.4).

Next we give the asymptotic behavior of $\Pi_0(u)$ as $u \rightarrow \infty$ based on an appropriate application of the Appendix in [5].

For any $\varepsilon_1 > 0$ and u large enough

$$\begin{aligned}
\Pi_0(u) &= \mathbb{P} \left\{ \sup_{t \in [S - \lambda u^{-2}, S]} \inf_{s \in [t, t + T_u]} X_u(s) > f_u(S) \right\} \\
&\leq \mathbb{P} \left\{ \sup_{t \in [S - \lambda u^{-2}, S]} \inf_{s \in [t, t + (1 - \varepsilon_1) T u^{-2}]} X_u(s) > f_u(S) \right\} \\
&= \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1 - \varepsilon_1) T]} X_u(S + u^{-2}s - u^{-2}t) > f_u(S) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1-\varepsilon_1)T]} Y_u(t, s) > f_u(S) \right\} \\
&=: \Pi_0^+(u)
\end{aligned}$$

and

$$\Pi_0(u) \geq \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1+\varepsilon_1)T]} Y_u(t, s) > f_u(S) \right\} =: \Pi_0^-(u),$$

where $Y_u(t, s) := X_u(S + u^{-2}s - u^{-2}t)$, for $(t, s) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$.

Since

$$\sigma_{Y_u}(t, s) := \sqrt{\text{Var}(Y_u(t, s))} = \sqrt{\text{Var}(X_u(S + u^{-2}s - u^{-2}t))} = \frac{f_u(S)}{f_u(S + u^{-2}s - u^{-2}t)}$$

and (3.4), there exists $d(t, s) = \frac{\delta e^{-2\delta S}}{1 - e^{-2\delta S}}(t - s)$ such that

$$(3.10) \quad \lim_{u \rightarrow \infty} \sup_{(t, s) \in [0, \lambda] \times [0, (1+\varepsilon_1)T]} |u^2(1 - \sigma_{Y_u}(t, s)) - d(t, s)| = 0.$$

Moreover, for $(t_1, s_1), (t_2, s_2) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$ and $s_1 - t_1 > s_2 - t_2$,

$$\begin{aligned}
&\text{Var}(Y_u(t_1, s_1) - Y_u(t_2, s_2)) \\
&= f_u^2(S) \mathbb{E} \left\{ \frac{X(S + u^{-2}s_1 - u^{-2}t_1)}{u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)})} - \frac{X(S + u^{-2}s_2 - u^{-2}t_2)}{u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_2 - u^{-2}t_2)})} \right\}^2 \\
&= f_u^2(S)(J_1(u) + J_2(u) + J_3(u)),
\end{aligned}$$

where

$$\begin{aligned}
J_1(u) &= \mathbb{E} \left\{ \frac{X(S + u^{-2}s_1 - u^{-2}t_1) - X(S + u^{-2}s_2 - u^{-2}t_2)}{u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)})} \right\}^2, \\
J_2(u) &= 2 \frac{\frac{\varepsilon}{\delta}(e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)} - e^{-\delta(S + u^{-2}s_2 - u^{-2}t_2)})}{(u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)}))(u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_2 - u^{-2}t_2)}))} \\
&\quad \times \mathbb{E} \left\{ \left(\frac{X(S + u^{-2}s_1 - u^{-2}t_1) - X(S + u^{-2}s_2 - u^{-2}t_2)}{u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)})} \right) X(S + u^{-2}s_2 - u^{-2}t_2) \right\} = 0, \\
J_3(u) &= \left(\frac{\frac{\varepsilon}{\delta}(e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)} - e^{-\delta(S + u^{-2}s_2 - u^{-2}t_2)})}{(u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)}))(u + \frac{\varepsilon}{\delta}(1 - e^{-\delta(S + u^{-2}s_2 - u^{-2}t_2)}))} \right)^2 \mathbb{E} \{ X(S + u^{-2}s_2 - u^{-2}t_2) \}^2.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{u \rightarrow \infty} u^2 f_u^2(S) J_1(u) &= \lim_{u \rightarrow \infty} f_u^2(S) \mathbb{E} \{ X(S + u^{-2}s_1 - u^{-2}t_1) - X(S + u^{-2}s_2 - u^{-2}t_2) \}^2 \\
&= \lim_{u \rightarrow \infty} \frac{u^2}{\frac{\sigma^2}{2\delta}(1 - e^{-2\delta S})} \frac{\sigma^2}{2\delta} (e^{-2\delta(S + u^{-2}s_2 - u^{-2}t_2)} - e^{-2\delta(S + u^{-2}s_1 - u^{-2}t_1)}) \\
&= \frac{2\delta e^{-2\delta S}}{1 - e^{-2\delta S}} ((s_1 - s_2) - (t_1 - t_2)) \\
&= \frac{2\delta e^{-2\delta S}}{1 - e^{-2\delta S}} \text{Var}(B(s_1 - t_1) - B(s_2 - t_2)), \\
\lim_{u \rightarrow \infty} u^2 f_u^2(S) J_3(u) &\leq \lim_{u \rightarrow \infty} \mathbb{C}_{11}(e^{-\delta(S + u^{-2}s_1 - u^{-2}t_1)} - e^{-\delta(S + u^{-2}s_2 - u^{-2}t_2)}) \mathbb{E} \{ X(S + u^{-2}s_2 - u^{-2}t_2) \}^2 = 0,
\end{aligned}$$

thus

$$(3.11) \quad \lim_{u \rightarrow \infty} u^2 \text{Var}(Y_u(t_1, s_1) - Y_u(t_2, s_2)) = \frac{2\delta e^{-2\delta S}}{1 - e^{-2\delta S}} \text{Var}(B(s_1 - t_1) - B(s_2 - t_2)).$$

Further, there exist some constant $G, u_0 > 0$, such that for any $u > u_0$

$$(3.12) \quad u^2 \text{Var}(Y_u(t_1, s_1) - Y_u(t_2, s_2)) \leq G(|t_1 - t_2| + |s_1 - s_2|)$$

holds uniformly with respect to $(t_1, s_1), (t_2, s_2) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$. By (3.10), (3.11), (3.12), Lemma 5.1 in [5] and $\lim_{u \rightarrow \infty} f_u(S)/u = 1/\sigma_X(S)$, we obtain

$$(3.13) \quad \Pi_0^-(u) \sim \tilde{\mathcal{P}}(a\lambda, a(1 + \varepsilon_1)T)\Psi(f_u(S)), \quad u \rightarrow \infty.$$

Similarly

$$\Pi_0^+(u) \sim \tilde{\mathcal{P}}(a\lambda, a(1 - \varepsilon_1)T)\Psi(f_u(S)), \quad u \rightarrow \infty.$$

Letting $\varepsilon_1 \rightarrow 0$ and $\lambda \rightarrow \infty$, we have

$$\Pi_0(u) \sim \tilde{\mathcal{P}}(aT)\Psi(f_u(S)), \quad u \rightarrow \infty.$$

The above combined with (3.3) and (3.9) drives (3.2), therefore by (3.1) the proof is complete. \square

PROOF OF THEOREM 2.3 Case 1 $\delta > 0$: According to the definition of conditional probability, for any $x, u > 0$

$$(3.14) \quad \begin{aligned} & \mathbb{P}\{u^2(S + T_u - \eta(u)) > x \mid \eta(u) \leq S + T_u\} \\ &= \frac{\mathbb{P}\left\{\sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \left(\sigma \int_0^s e^{-\delta z} dB(z) - c \int_0^s e^{-\delta z} dz\right) > u\right\}}{\mathbb{P}\left\{\sup_{t \in [0, S]} \inf_{s \in [t, t+T_u]} \left(\sigma \int_0^s e^{-\delta z} dB(z) - c \int_0^s e^{-\delta z} dz\right) > u\right\}}. \end{aligned}$$

Using the same notation of $X(s)$, $\bar{X}(s)$, $f_u(s)$, $X_u(s)$, $\sigma_X(s)$ as in the proof of Theorem 2.1, we have for u large enough

$$\begin{aligned} & \mathbb{P}\left\{\sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \left(\sigma \int_0^s e^{-\delta z} dB(z) - c \int_0^s e^{-\delta z} dz\right) > u\right\} \\ &= \mathbb{P}\left\{\sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \bar{X}(s) \frac{f_u(S)}{f_u(s)} > f_u(S)\right\} \\ &= \mathbb{P}\left\{\sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} X_u(s) > f_u(S)\right\}, \end{aligned}$$

Set $\rho(u) = \left(\frac{\ln u}{u}\right)^2$. For any $\lambda > 0$, Bonferroni inequality yields

$$(3.15) \quad \Pi_0^*(u) \leq \mathbb{P}\left\{\sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} X_u(s) > f_u(S)\right\} \leq \Pi_0^*(u) + \Pi_1^*(u) + \Pi_2^*(u),$$

where

$$\begin{aligned} \Pi_0^*(u) &= \mathbb{P}\left\{\sup_{t \in [S-xu^{-2}-\lambda u^{-2}, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} X_u(s) > f_u(S)\right\}, \\ \Pi_1^*(u) &= \mathbb{P}\left\{\sup_{t \in [0, S-\rho(u)]} \inf_{s \in [t, t+T_u]} X_u(s) > f_u(S)\right\}, \\ \Pi_2^*(u) &= \mathbb{P}\left\{\sup_{t \in [S-\rho(u), S-xu^{-2}-\lambda u^{-2}]} \inf_{s \in [t, t+T_u]} X_u(s) > f_u(S)\right\}. \end{aligned}$$

By (3.3) and (3.9) in the proof of Theorem 2.1, we know

$$(3.16) \quad \Pi_1^*(u) = o(\Psi(f_u(S))), \quad u \rightarrow \infty,$$

and

$$(3.17) \quad \Pi_2^*(u) \leq \mathbb{P} \left\{ \sup_{t \in [S - \rho(u), S - \lambda u^{-2}]} \inf_{s \in [t, t + T_u]} X_u(s) > f_u(S) \right\} = o(\Psi(f_u(S))), \quad u \rightarrow \infty, \quad \lambda \rightarrow \infty.$$

Next we give the asymptotic behavior of $\Pi_0^*(u)$ as $u \rightarrow \infty$. For any $\varepsilon_1 > 0$ and u large enough

$$\begin{aligned} \Pi_0^*(u) &= \mathbb{P} \left\{ \sup_{t \in [S - xu^{-2} - \lambda u^{-2}, S - xu^{-2}]} \inf_{s \in [t, t + T_u]} \overline{X}(s) \frac{f_u(S)}{f_u(s)} > f_u(S) \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in [S - xu^{-2} - \lambda u^{-2}, S - xu^{-2}]} \inf_{s \in [t, t + T_u]} \overline{X}(s) \frac{f_u(S - xu^{-2})}{f_u(s)} > f_u(S - xu^{-2}) \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in [S - xu^{-2} - \lambda u^{-2}, S - xu^{-2}]} \inf_{s \in [t, t + (1 - \varepsilon_1)T_u]} \overline{X}(s) \frac{f_u(S - xu^{-2})}{f_u(s)} > f_u(S - xu^{-2}) \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1 - \varepsilon_1)T]} \overline{X}(S + u^{-2}s - u^{-2}t - u^{-2}x) \frac{f_u(S - xu^{-2})}{f_u(S + u^{-2}s - u^{-2}t - u^{-2}x)} > f_u(S - xu^{-2}) \right\} \\ &= \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1 - \varepsilon_1)T]} Y_u^*(t, s) > f_u(S - xu^{-2}) \right\} \\ &=: \Pi_0^{*+}(u), \end{aligned}$$

and

$$\Pi_0^*(u) \geq \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1 + \varepsilon_1)T]} Y_u^*(t, s) > f_u(S - xu^{-2}) \right\} =: \Pi_0^{*-}(u),$$

where $Y_u^*(t, s) := \overline{X}(S + u^{-2}s - u^{-2}t - u^{-2}x) \frac{f_u(S - xu^{-2})}{f_u(S + u^{-2}s - u^{-2}t - u^{-2}x)}$, $(t, s) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$ and $\sigma_{Y_u^*}^2(t, s) := \text{Var}(Y_u^*(t, s)) = \left(\frac{f_u(S - xu^{-2})}{f_u(S + u^{-2}s - u^{-2}t - u^{-2}x)} \right)^2$.

Using the similar argumentation as (3.10) in the proof of Theorem 2.1, we have

$$(3.18) \quad \lim_{u \rightarrow \infty} \sup_{(t, s) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]} |u^2(1 - \sigma_{Y_u^*}(t, s)) - d(t, s)| = 0,$$

with $d(t, s) = \frac{\delta e^{-2\delta S}}{1 - e^{-2\delta S}}(t - s)$. Moreover, (3.11), (3.12) still hold for $Y_u^*(t, s)$ and $(t_1, s_1), (t_2, s_2) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$. By Lemma 5.1 in [5] and $\lim_{u \rightarrow \infty} f_u(S)/u = 1/\sigma_X(S)$, we obtain

$$\Pi_0^{*-}(u) \sim \tilde{\mathcal{P}}(a\lambda, a(1 + \varepsilon_1)T) \Psi(f_u(S - xu^{-2})) \sim e^{-ax} \tilde{\mathcal{P}}(a\lambda, a(1 + \varepsilon_1)T) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

Similarly,

$$\Pi_0^{*+}(u) \sim e^{-ax} \tilde{\mathcal{P}}(a\lambda, a(1 - \varepsilon_1)T) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

Letting $\varepsilon_1 \rightarrow 0$ and $\lambda \rightarrow \infty$, we have

$$\Pi_0^*(u) \sim e^{-ax} \tilde{\mathcal{P}}(aT) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

The above combined with (3.15), (3.16) and (3.17) derives that

$$\mathbb{P} \left\{ \sup_{t \in [0, S - xu^{-2}]} \inf_{s \in [t, t + T_u]} X_u(s) > f_u(S) \right\} \sim e^{-ax} \tilde{\mathcal{P}}(aT) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

Thus, the claim follows by using the results of Theorem 2.1 and (3.14).

Case 2 $\delta = 0$:

$$\mathbb{P} \{ u^2(S + T_u - \eta(u)) > x | \eta(u) \leq S + T_u \} = \frac{\mathbb{P} \left\{ \sup_{t \in [0, S - xu^{-2}]} \inf_{s \in [t, t + T_u]} (\sigma B(s) - cs) > u \right\}}{\mathbb{P} \left\{ \sup_{t \in [0, S]} \inf_{s \in [t, t + T_u]} (\sigma B(s) - cs) > u \right\}}.$$

For u large enough

$$\mathbb{P} \left\{ \sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} (\sigma B(s) - cs) > u \right\} = \mathbb{P} \left\{ \sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \tilde{X}_u(s) > f_u(S) \right\},$$

with

$$X(s) = \sigma B(s), \quad \overline{X}(s) = \frac{B(s)}{\sqrt{s}}, \quad f_u(s) = \frac{u + cs}{\sigma \sqrt{s}} \quad \text{and} \quad \tilde{X}_u(s) = \overline{X}(s) \frac{f_u(S)}{f_u(s)}.$$

Set $\rho(u) = \left(\frac{\ln u}{u}\right)^2$. For any $\lambda > 0$, Bonferroni inequality yields

$$(3.19) \quad \tilde{\Pi}_0(u) \leq \mathbb{P} \left\{ \sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \tilde{X}_u(s) > f_u(S) \right\} \leq \tilde{\Pi}_0(u) + \tilde{\Pi}_1(u) + \tilde{\Pi}_2(u),$$

where

$$\begin{aligned} \tilde{\Pi}_0(u) &= \mathbb{P} \left\{ \sup_{t \in [S-xu^{-2}-\lambda u^{-2}, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \tilde{X}_u(s) > f_u(S) \right\}, \\ \tilde{\Pi}_1(u) &= \mathbb{P} \left\{ \sup_{t \in [0, S-\rho(u)]} \inf_{s \in [t, t+T_u]} \tilde{X}_u(s) > f_u(S) \right\}, \\ \tilde{\Pi}_2(u) &= \mathbb{P} \left\{ \sup_{t \in [S-\rho(u), S-xu^{-2}-\lambda u^{-2}]} \inf_{s \in [t, t+T_u]} \tilde{X}_u(s) > f_u(S) \right\}. \end{aligned}$$

Notice that for u large enough

$$\begin{aligned} \mathbb{E} \left\{ (\tilde{X}_u(t_1) - \tilde{X}_u(t_2))^2 \right\} &= \frac{1}{S} \mathbb{E} \left\{ \left(\frac{u + cS}{u + ct_1} B(t_1) - \frac{u + cS}{u + ct_2} B(t_2) \right)^2 \right\} \\ &\leq \mathbb{C}_{12} \mathbb{E} \left\{ (B(t_1) - B(t_2))^2 \right\} + \mathbb{C}_{13} \left(\frac{u + cS}{u + ct_1} - \frac{u + cS}{u + ct_2} \right)^2 \\ &\leq \mathbb{C}_{14} |t_1 - t_2|, \quad t_1 < t_2, \quad t_1, t_2 \in (0, S], \end{aligned}$$

and

$$\sup_{t \in [0, S-\rho(u)]} \text{Var} \left(\tilde{X}_u(t) \right) = \sup_{t \in [0, S-\rho(u)]} \left(\frac{f_u(S)}{f_u(t)} \right)^2 = \frac{f_u^2(S)}{f_u^2(S-\rho(u))},$$

where we use the fact that $f_u(t)$ is a decreasing function for $t \in [0, S]$ when u large enough.

Moreover,

$$1 - \frac{f_u(S)}{f_u(S-t)} \sim \frac{1}{2S} t, \quad t \rightarrow 0,$$

$$\inf_{1 \leq k \leq N(u)} f_u(S - k\lambda u^{-2}) \rightarrow \infty, \quad u \rightarrow \infty,$$

and for $t_1 < t_2$, $t_1, t_2 \in [0, S]$,

$$r_{\tilde{X}}(t_1, t_2) := \mathbb{E} \left\{ \overline{X}(t_1) \overline{X}(t_2) \right\} = \sqrt{\frac{t_1}{t_2}}.$$

Then

$$\begin{aligned} (3.20) \quad & \lim_{u \rightarrow \infty} \sup_{1 \leq k \leq N(u)} \sup_{\substack{t_1 \neq t_2, \\ t_1, t_2 \in [0, \lambda]}} \left| \frac{f_u^2(S - k\lambda u^{-2}) \text{Var}(\overline{X}(S - u^{-2}t_1) - \overline{X}(S - u^{-2}t_2))}{2b|t_1 - t_2|} - 1 \right| \\ &= \lim_{u \rightarrow \infty} \sup_{1 \leq k \leq N(u)} \sup_{\substack{t_1 \neq t_2, \\ t_1, t_2 \in [0, \lambda]}} \left| \frac{f_u^2(S - k\lambda u^{-2}) (2 - 2r_{\tilde{X}}(S - u^{-2}t_1, S - u^{-2}t_2))}{2b|t_1 - t_2|} - 1 \right| = 0, \end{aligned}$$

where $b = \frac{1}{2\sigma^2 S^2}$, and

$$\begin{aligned}
& \sup_{1 \leq k \leq N(u)} \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} f_u^2(S - k\lambda u^{-2}) \mathbb{E} \{ (\overline{X}(S - u^{-2}t_1) - \overline{X}(S - u^{-2}t_2)) \overline{X}(S) \} \\
& \leq \mathbb{C}_{15} u^2 \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} |r_{\tilde{X}}(S - u^{-2}t_1, S) - r_{\tilde{X}}(S - u^{-2}t_2, S)| \\
& \leq \mathbb{C}_{16} u^2 \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} \left| \sqrt{S - u^{-2}t_1} - \sqrt{S - u^{-2}t_2} \right| \\
(3.21) \quad & \leq \mathbb{C}_{17} \sup_{\substack{|t_1 - t_2| < \varepsilon \\ t_1, t_2 \in [0, \lambda]}} |t_1 - t_2| \rightarrow 0, \quad u \rightarrow \infty, \varepsilon \rightarrow 0.
\end{aligned}$$

By Theorem 8.1 in [17] and Lemma 5.3 in [2], using the similar argumentation as in the proof of Theorem 2.1, we derive

$$(3.22) \quad \tilde{\Pi}_1(u) + \tilde{\Pi}_2(u) = o(\Psi(f_u(S))), \quad u \rightarrow \infty, \lambda \rightarrow \infty.$$

Next we give the asymptotic behavior of $\tilde{\Pi}_0(u)$ as $u \rightarrow \infty$. For any $\varepsilon_1 > 0$ and u large enough

$$\begin{aligned}
\tilde{\Pi}_0(u) &= \mathbb{P} \left\{ \sup_{t \in [S - xu^{-2} - \lambda u^{-2}, S - xu^{-2}]} \inf_{s \in [t, t + T_u]} \overline{X}(s) \frac{f_u(S)}{f_u(s)} > f_u(S) \right\} \\
&= \mathbb{P} \left\{ \sup_{t \in [S - xu^{-2} - \lambda u^{-2}, S - xu^{-2}]} \inf_{s \in [t, t + T_u]} \overline{X}(s) \frac{f_u(S - xu^{-2})}{f_u(s)} > f_u(S - xu^{-2}) \right\} \\
&\leq \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1 - \varepsilon_1)T]} \tilde{Y}_u(t, s) > f_u(S - xu^{-2}) \right\} \\
&=: \tilde{\Pi}_0^+(u)
\end{aligned}$$

and

$$\tilde{\Pi}_0(u) \geq \mathbb{P} \left\{ \sup_{t \in [0, \lambda]} \inf_{s \in [0, (1 + \varepsilon_1)T]} \tilde{Y}_u(t, s) > f_u(S - xu^{-2}) \right\} =: \tilde{\Pi}_0^-(u),$$

where $\tilde{Y}_u(t, s) := \overline{X}(S + u^{-2}s - u^{-2}t - u^{-2}x) \frac{f_u(S - xu^{-2})}{f_u(S + u^{-2}s - u^{-2}t - u^{-2}x)}$, for $(t, s) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$.

Using the similar argumentation as (3.10), (3.11) and (3.12) in the proof of Theorem 2.1, we obtain that

$$(3.23) \quad \lim_{u \rightarrow \infty} \sup_{(t, s) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]} \left| u^2(1 - \sigma_{\tilde{Y}_u}(t, s)) - \tilde{d}(t, s) \right| = 0,$$

with $\tilde{d}(t, s) = \frac{1}{2S}(t - s)$ and $\sigma_{\tilde{Y}_u}(t, s) := \sqrt{\text{Var}(\tilde{Y}_u(t, s))}$,

$$\lim_{u \rightarrow \infty} u^2 \text{Var}(\tilde{Y}_u(t_1, s_1) - \tilde{Y}_u(t_2, s_2)) = \frac{1}{S} \text{Var}(B(s_1 - t_1) - B(s_2 - t_2)),$$

and for some constant G and all u large enough

$$u^2 \text{Var}(\tilde{Y}_u(t_1, s_1) - \tilde{Y}_u(t_2, s_2)) \leq G(|t_1 - t_2| + |s_1 - s_2|)$$

uniformly for $(t_1, s_1), (t_2, s_2) \in [0, \lambda] \times [0, (1 + \varepsilon_1)T]$.

By Lemma 5.1 in [5] and $\lim_{u \rightarrow \infty} f_u(S)/u = \frac{1}{\sigma\sqrt{S}}$, we obtain

$$\tilde{\Pi}_0^-(u) \sim \tilde{\mathcal{P}}(b\lambda, b(1 + \varepsilon_1)T) \Psi(f_u(S - xu^{-2})) \sim e^{-bx} \tilde{\mathcal{P}}(b\lambda, b(1 + \varepsilon_1)T) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

Similarly,

$$\tilde{\Pi}_0^+(u) \sim e^{-bx} \tilde{\mathcal{P}}(b\lambda, b(1 - \varepsilon_1)T) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

Letting $\varepsilon_1 \rightarrow 0$ and $\lambda \rightarrow \infty$, we have

$$\tilde{\Pi}_0(u) \sim e^{-bx} \tilde{\mathcal{P}}(bT) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

The above combined with (3.19) and (3.22) leads to

$$\mathbb{P} \left\{ \sup_{t \in [0, S-xu^{-2}]} \inf_{s \in [t, t+T_u]} \tilde{X}_u(s) > f_u(S_x(u)) \right\} \sim e^{-bx} \tilde{\mathcal{P}}(bT) \Psi(f_u(S)), \quad u \rightarrow \infty.$$

Using the above asymptotic equality and b) of Remarks 2.2, we obtain the results. □

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